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A Class of Polynomials in Two Variables

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Abstract. In this paper, we present some families of polynomials in two variables. Some further results of these polynomials as generating function, Rodrigues formula and recurrence relations are discussed. We derive various families of bilinear and bilateral generating functions. We also give some particular cases reduced to Hermite-Hermite and Laguerre-Laguerre polynomials.

1. Introduction

Recently, several families of multivariable polynomials have been derived in [2,3,4]. In this paper, we give some families of polynomials in two variables. In [1], we presented a family of polynomials defined through Rodrigues formula:

$$\phi_{k+n(m-1)}\left(x\right)=e^{\varphi_m(x)}\frac{d^n}{dx^n}\left(\psi_k\left(x\right)e^{-\varphi_m(x)}\right) \tag{1.1}$$

where $\phi_{k+n(m-1)}\left(x\right)$ is a polynomial of degree $k+n\left(m-1\right)$, $n=0,1,2,\ldots$ and, $\psi_k\left(x\right)$ and $\varphi_m\left(x\right)$ are polynomials respectively of degree k and m; $k,m=0,1,2,\ldots$. In that work, for these polynomials whose special cases are reduced to Hermite polynomials, we gave some recurrence relations and generating function. In this paper, as a generalization of polynomials (1.1), we present a family of polynomials in two variables and obtain some relations satisfied by these polynomials. We also derive various families of bilinear and bilateral generating functions for these polynomials.

Krall and Sheffer [5] (see also [8]) showed that the products of two classical orthogonal polynomial are orthogonal. Two of such polynomials are respectively Hermite-Hermite and Laguerre-Laguerre polynomials which are of degree $m+n$

$$F_{n+m,m}\left(x,y\right)=H_n\left(x\right)H_m\left(y\right);\quad m,n=0,1,\ldots$$

and

$$K_{n+m,m}\left(x,y\right)=L_n\left(x\right)L_m\left(y\right);\quad m,n=0,1,\ldots$$

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Hermite-Hermite polynomials are orthogonal over the domain

$$D = \{(x, y) : -\infty < x < \infty,\ -\infty < y < \infty\}$$

with respect to the weight function $\rho(x, y) = e^{-x^2-y^2}$. Similarly, Laguerre-Laguerre polynomials are orthogonal with respect to the weight function $\rho(x, y) = e^{-x-y}$ over the domain

$$D = \{(x, y) : x > 0,\ y > 0\}.$$

We define a family of polynomials whose particular cases are reduced to Hermite-Hermite and Laguerre-Laguerre polynomials with Rodrigues formula:

$$\phi_{N,\ n_2}(x, y) = e^{\varphi_m(x,y)} \frac{\partial^{n_1+n_2}}{\partial x^{n_1} \partial y^{n_2}} \left\{ \psi_k(x, y)\, e^{-\varphi_m(x,y)} \right\} \tag{1.2}$$

$$(n_1, n_2, k, m = 0, 1, 2, \ldots,)$$

where $N = (m-1)(n_1+n_2)+k$ denotes the total degree of the polynomial with respect to the variables x and y, and subindex n_2 gives the order of polynomials which are of degree $(m-1)(n_1+n_2)+k$. Let $\psi_k(x, y)$ and $\varphi_m(x, y)$ be polynomials of respectively total degree k and m with respect to x and y. Also, let $\varphi_m(x, y)$ contain at least two of monomials $\left\{x^{m-k}y^k\ ,\ k = 0, 1, \ldots, m\right\}$ of degree m. Taking $n_1 - n_2$ instead of n_1 in (1.2), the polynomials (1.2) give (n_1+1) polynomials which are of degree $(m-1)n_1+k$ for $n_2 = 0, 1, \ldots, n_1$.

The main object of this paper is to construct a polynomial set which contain Hermite-Hermite and Laguerre-Laguerre orthogonal polynomials and to give some relations satisfied by these polynomials. We derive various families of bilinear and bilateral generating functions for the polynomials $\phi_{N,\ n_2}(x, y)$ given by (1.2). We also show that some particular cases of (1.2) are reduced to Hermite-Hermite and Laguerre-Laguerre orthogonal polynomials.

2. Generating Function for the Polynomials $\phi_{N,\ n_2}(x, y)$

In this section, we give some families of generating function for the polynomials $\phi_{N,\ n_2}(x, y)$ with two variables in the form

$$F(x, y; r_1, r_2) = \sum_{n_1, n_2=0}^{\infty} a_{n_1,n_2} P_{n_1,n_2}(x, y)\ r_1^{n_1} r_2^{n_2}$$

with the help of the Cauchy's integral formula.

Theorem 2.1. For the polynomials $\phi_{N,\ n_2}(x, y)$, we have

$$\sum_{n_1,\ n_2=0}^{\infty} \phi_{N,\ n_2}(x, y) \frac{r_1^{n_1}}{n_1!} \frac{r_2^{n_2}}{n_2!} = F(x, y; r_1, r_2) \tag{2.1}$$

where

$$F(x,y;r_1,r_2)=\psi_k(x+r_1,y+r_2)\,e^{\varphi_m(x,y)-\varphi_m(x+r_1,y+r_2)}.$$

Proof. Consider the series

$$A_{n_2}(x,y,r_1):=\sum_{n_1=0}^{\infty}\phi_{N,\ n_2}(x,y)\frac{r_1^{n_1}}{n_1!}.$$

Since

$$\phi_{N,\ n_2}(x,y)=e^{\varphi_m(x,y)}\frac{\partial^{n_1+n_2}}{\partial x^{n_1}\partial y^{n_2}}\left\{\psi_k(x,y)\,e^{-\varphi_m(x,y)}\right\},$$

with the help of the Cauchy's integral formula,we can write that

$$\begin{aligned}A_{n_2}(x,y,r_1) &= e^{\varphi_m(x,y)}\sum_{n_1=0}^{\infty}\frac{\partial^{n_2}}{\partial y^{n_2}}\left\{\frac{1}{2\pi i}\oint\limits_{C_1}\frac{\psi_k(z,y)\,e^{-\varphi_m(z,y)}}{(z-x)^{n_1+1}}dz\right\}r_1^{n_1}\\ &= \frac{e^{\varphi_m(x,y)}}{2\pi i}\frac{\partial^{n_2}}{\partial y^{n_2}}\oint\limits_{C_1}\frac{\psi_k(z,y)\,e^{-\varphi_m(z,y)}}{z-x}\sum_{n_1=0}^{\infty}\left(\frac{r_1}{z-x}\right)^{n_1}dz\end{aligned}$$

where the closed contour C_1 in the complex $z-$plane is a circle (centered at $z=x$) of sufficiently small radius, which is described in the positive direction (counter-clockwise). Therefore

$$\begin{aligned}A_{n_2}(x,y,r_1) &= \frac{e^{\varphi_m(x,y)}}{2\pi i}\frac{\partial^{n_2}}{\partial y^{n_2}}\oint\limits_{C_2}\frac{\psi_k(z,y)\,e^{-\varphi_m(z,y)}}{z-(x+r_1)}dz\ ,\left|\frac{r_1}{z-x}\right|<1\\ &= e^{\varphi_m(x,y)}\frac{\partial^{n_2}}{\partial y^{n_2}}\left\{\psi_k(x+r_1,y)\,e^{-\varphi_m(x+r_1,y)}\right\}\end{aligned}$$

where C_2 is a circle (centered at $z=x+r_1$) with radius $\varepsilon>0$ in the complex $z-$plane. Thus

$$\sum_{n_1=0}^{\infty}\phi_{N,\ n_2}(x,y)\frac{r_1^{n_1}}{n_1!}=e^{\varphi_m(x,y)}\frac{\partial^{n_2}}{\partial y^{n_2}}\left\{\psi_k(x+r_1,y)\,e^{-\varphi_m(x+r_1,y)}\right\}. \tag{2.2}$$

Multiplying both sides of (2.2) by $\dfrac{r_2^{n_2}}{n_2!}$ and then summing both sides, we obtain

$$\begin{aligned}&\sum_{n_1,\ n_2=0}^{\infty}\phi_{N,\ n_2}(x,y)\frac{r_1^{n_1}}{n_1!}\frac{r_2^{n_2}}{n_2!}\\ &=\sum_{n_2=0}^{\infty}e^{\varphi_m(x,y)}\tfrac{\partial^{n_2}}{\partial y^{n_2}}\left\{\psi_k(x+r_1,y)\,e^{-\varphi_m(x+r_1,y)}\right\}\frac{r_2^{n_2}}{n_2!}.\end{aligned} \tag{2.3}$$

Applying the Cauchy's integral formula again to the right hand-side of (2.3), for suitable contour C_3, we have

$$\sum_{n_1,\ n_2=0}^{\infty} \phi_{N,\ n_2}\left(x,y\right) \frac{r_1^{n_1}}{n_1!}\frac{r_2^{n_2}}{n_2!}$$

$$= \frac{e^{\varphi_m(x,y)}}{2\pi i} \oint\limits_{C_3} \frac{\psi_k\left(x+r_1,\eta\right)e^{-\varphi_m(x+r_1,\eta)}}{\eta-\left(y+r_2\right)}d\eta, \left|\frac{r_2}{\eta-y}\right| < 1$$
$$= \psi_k\left(x+r_1,y+r_2\right)e^{\varphi_m(x,y)-\varphi_m(x+r_1,y+r_2)}$$

which completes the proof of the Theorem 2.1.

3. Some Recurrence Relations of the Polynomials $\phi_{N,\ n_2}\left(x,y\right)$

In this section, we give some recurrence relations satisfied by the polynomial set $\phi_{N,\ n_2}\left(x,y\right)$ for some special choices of the polynomials $\psi_k\left(x,y\right)$. Choosing $\psi_k\left(x,y\right)=\left(ax+by+c\right)^k$, $(a,b,c\in\mathbb{R}$, $k=0,1,\ldots)$ in (1.2), we get the polynomials:

$$\phi_{N,\ n_2}\left(x,y\right) = e^{\varphi_m(x,y)}\frac{\partial^{n_1+n_2}}{\partial x^{n_1}\partial y^{n_2}}\left\{\left(ax+by+c\right)^k e^{-\varphi_m(x,y)}\right\}. \tag{3.1}$$

As a consequence of Theorem 2.1, the generating function for the polynomials (3.1) is given by

$$\sum_{n_1,\ n_2=0}^{\infty} \phi_{N,\ n_2}\left(x,y\right) \frac{r_1^{n_1}}{n_1!}\frac{r_2^{n_2}}{n_2!} = F\left(x,y;r_1,r_2\right) \tag{3.2}$$

where

$$F\left(x,y;r_1,r_2\right) = e^{\varphi_m(x,y)-\varphi_m(x+r_1,y+r_2)}\left[a\left(x+r_1\right)+b\left(y+r_2\right)+c\right]^k.$$

For these polynomials given by (3.1), we have the following results.

Theorem 3.1. Let

$$\Omega^{(p,l)}\left(x,y;n_1,n_2\right) = \left(ax+by+c\right)\binom{n_1}{l}\binom{n_2}{p-l}\phi_{N-pm+p,\ n_2+l-p}\left(x,y\right)$$
$$+a\left(l+1\right)\binom{n_1}{l+1}\binom{n_2}{p-l}\phi_{N-(p+1)(m-1),\ n_2+l-p}\left(x,y\right)$$
$$+b\left(p-l+1\right)\binom{n_1}{l}\binom{n_2}{p-l+1}\phi_{N-(p+1)(m-1),\ n_2+l-p-1}\left(x,y\right).$$

Then for the polynomials $\phi_{N,\ n_2}(x,y)$, we have the following recurrence relations

$$-\sum_{p=0}^{m-1}\sum_{l=0}^{p}\Omega^{(p,l)}(x,y;n_1,n_2)\frac{\partial^{p+1}}{\partial x^{l+1}\partial y^{p-l}}\varphi_m(x,y)$$
$$=(ax+by+c)\,\phi_{N+m-1,\ n_2}(x,y)+a(n_1-k)\,\phi_{N,\ n_2}(x,y)$$
$$+bn_2\phi_{N,\ n_2-1}(x,y)$$

and

$$-\sum_{p=0}^{m-1}\sum_{l=0}^{p}\Omega^{(p,l)}(x,y;n_1,n_2)\frac{\partial^{p+1}}{\partial x^{l}\partial y^{p-l+1}}\varphi_m(x,y)$$
$$=\ (ax+by+c)\,\phi_{N+m-1,\ n_2+1}(x,y)+an_1\phi_{N,\ n_2+1}(x,y)$$
$$+b(n_2-k)\,\phi_{N,\ n_2}(x,y)$$

where

$$m\geq 1\ ,\ n_1\geq l+1\ ,\ n_2\geq p-l+1.$$

Proof. The Taylor series of the polynomial $\frac{\partial}{\partial r_1}\varphi_m(x+r_1,y+r_2)$ at $(r_1,r_2)=(0,0)$ is given with

$$\frac{\partial}{\partial r_1}\varphi_m(x+r_1,y+r_2)=\sum_{p=0}^{m-1}\sum_{l=0}^{p}\frac{1}{p!}\binom{p}{l}\frac{\partial^{p+1}}{\partial x^{l+1}\partial y^{p-l}}\varphi_m(x,y)\,r_1^l r_2^{p-l}. \tag{3.3}$$

Differentiating each member of the generating function (3.2) with respect to r_1 and using (3.2), we find that

$$\sum_{n_1,\ n_2=0}^{\infty}\phi_{N,\ n_2}(x,y)\frac{n_1r_1^{n_1-1}}{n_1!}\frac{r_2^{n_2}}{n_2!}$$
$$=\frac{ak}{a(x+r_1)+b(y+r_2)+c}\sum_{n_1,\ n_2=0}^{\infty}\phi_{N,\ n_2}(x,y)\frac{r_1^{n_1}}{n_1!}\frac{r_2^{n_2}}{n_2!}+$$
$$+\frac{\partial}{\partial r_1}\varphi_m(x+r_1,y+r_2)\sum_{n_1,\ n_2=0}^{\infty}\phi_{N,\ n_2}(x,y)\frac{r_1^{n_1}}{n_1!}\frac{r_2^{n_2}}{n_2!}.$$

Using (3.3) in the last equality, therefore we get the first desired recurrence relation for the polynomials $\phi_{N,\ n_2}(x,y)$.

On the other hand, the Taylor series of the polynomial $\frac{\partial}{\partial r_2}\varphi_m(x+r_1,y+r_2)$ at $(r_1,r_2)=(0,0)$ is

$$\frac{\partial}{\partial r_2}\varphi_m(x+r_1,y+r_2)=\sum_{p=0}^{m-1}\sum_{l=0}^{p}\frac{1}{p!}\binom{p}{l}\frac{\partial^{p+1}}{\partial x^{l}\partial y^{p-l+1}}\varphi_m(x,y)\,r_1^l r_2^{p-l}. \tag{3.4}$$

Similarly, if we differentiate two hand side of (3.2) with respect to r_2 and use (3.2) and (3.4), we obtain the second recurrence relation.

Other recurrence relations for the polynomials $\phi_{N,\ n_2}(x,y)$ can be obtained by differentiating the generating function (3.2) with respect to x and y, immediately.

Theorem 3.2. Let $\Omega^{(p,l)}(x,y;n_1,n_2)$ be as in Theorem 3.1. Then for the polynomials $\phi_{N,\ n_2}(x,y)$, we have the following recurrence relations

$$\begin{aligned} &-\sum_{p=0}^{m-1}\sum_{l=0}^{p}\Omega^{(p,l)}(x,y;n_1,n_2)\frac{\partial^{p+1}}{\partial x^{l+1}\partial y^{p-l}}\varphi_m(x,y) \\ = \;& (ax+by+c)\frac{\partial}{\partial x}\phi_{N,\ n_2}(x,y)+an_1\frac{\partial}{\partial x}\phi_{N-m+1,\ n_2}(x,y) \\ &+bn_2\frac{\partial}{\partial x}\phi_{N-m+1,\ n_2-1}(x,y)-an_1\phi_{N-m+1,\ n_2}(x,y)\frac{\partial}{\partial x}\varphi_m(x,y) \\ &-\left\{ak+(ax+by+c)\frac{\partial}{\partial x}\varphi_m(x,y)\right\}\phi_{N,\ n_2}(x,y) \\ &-bn_2\phi_{N-m+1,\ n_2-1}(x,y)\frac{\partial}{\partial x}\varphi_m(x,y) \end{aligned}$$

and

$$\begin{aligned} &-\sum_{p=0}^{m-1}\sum_{l=0}^{p}\Omega^{(p,l)}(x,y;n_1,n_2)\frac{\partial^{p+1}}{\partial x^{l}\partial y^{p-l+1}}\varphi_m(x,y) \\ = \;& (ax+by+c)\frac{\partial}{\partial y}\phi_{N,\ n_2}(x,y)+an_1\frac{\partial}{\partial y}\phi_{N-m+1,\ n_2}(x,y) \\ &+bn_2\frac{\partial}{\partial y}\phi_{N-m+1,\ n_2-1}(x,y)-an_1\phi_{N-m+1,\ n_2}(x,y)\frac{\partial}{\partial y}\varphi_m(x,y) \\ &-\left\{bk+(ax+by+c)\frac{\partial}{\partial y}\varphi_m(x,y)\right\}\phi_{N,\ n_2}(x,y) \\ &-bn_2\phi_{N-m+1,\ n_2-1}(x,y)\frac{\partial}{\partial y}\varphi_m(x,y) \end{aligned}$$

where

$$m\geq 1,\;\; n_1\geq l+1,\;\;\; n_2\geq p-l+1.$$

The next results can be easily obtained from Theorem 3.1 and Theorem 3.2.

Theorem 3.3. For the polynomials $\phi_{N,\ n_2}(x,y)$, we get as follows

$$\begin{aligned}
& (ax+by+c)\,\phi_{N+m-1,\ n_2}(x,y) \\
& +\left\{an_1+(ax+by+c)\frac{\partial}{\partial x}\varphi_m(x,y)\right\}\phi_{N,\ n_2}(x,y) \\
= & \ (ax+by+c)\frac{\partial}{\partial x}\phi_{N,\ n_2}(x,y)+an_1\frac{\partial}{\partial x}\phi_{N-m+1,\ n_2}(x,y) \\
& +bn_2\frac{\partial}{\partial x}\phi_{N-m+1,\ n_2-1}(x,y)-an_1\phi_{N-m+1,\ n_2}(x,y)\frac{\partial}{\partial x}\varphi_m(x,y) \\
& -bn_2\left\{\phi_{N-m+1,\ n_2-1}(x,y)\frac{\partial}{\partial x}\varphi_m(x,y)+\phi_{N,\ n_2-1}(x,y)\right\}
\end{aligned}$$

and

$$\begin{aligned}
& (ax+by+c)\,\phi_{N+m-1,\ n_2+1}(x,y) \\
& +\left\{bn_2+(ax+by+c)\frac{\partial}{\partial y}\varphi_m(x,y)\right\}\phi_{N,\ n_2}(x,y) \\
= & \ (ax+by+c)\frac{\partial}{\partial y}\phi_{N,\ n_2}(x,y)+bn_2\frac{\partial}{\partial y}\phi_{N-m+1,\ n_2-1}(x,y) \\
& +an_1\frac{\partial}{\partial y}\phi_{N-m+1,\ n_2}(x,y)-bn_2\phi_{N-m+1,\ n_2-1}(x,y)\frac{\partial}{\partial y}\varphi_m(x,y) \\
& -an_1\left\{\phi_{N-m+1,\ n_2}(x,y)\frac{\partial}{\partial y}\varphi_m(x,y)+\phi_{N,\ n_2+1}(x,y)\right\}.
\end{aligned}$$

Setting $\psi_k(x,y)=1$ in (1.2), we obtain the polynomials

$$\phi_{N_1,\ n_2}(x,y)=e^{\varphi_m(x,y)}\frac{\partial^{n_1+n_2}}{\partial x^{n_1}\partial y^{n_2}}\left\{e^{-\varphi_m(x,y)}\right\}. \tag{3.5}$$

which are of degree $N_1=(m-1)(n_1+n_2)$. As a result of Theorem 3.1-3.3, we have following:

Corollary 3.4. The polynomials $\phi_{N_1,\ n_2}(x,y)$ hold respectively:

$$\begin{aligned}
& \phi_{N_1+m-1,\ n_2}(x,y) \\
= & -\sum_{p=0}^{m-1}\sum_{l=0}^{p}\binom{n_1}{l}\binom{n_2}{p-l}\phi_{N_1-pm+p,\ n_2+l-p}(x,y)\frac{\partial^{p+1}}{\partial x^{l+1}\partial y^{p-l}}\varphi_m(x,y),
\end{aligned}$$

$$\begin{aligned}
& \phi_{N_1+m-1,\ n_2+1}(x,y) \\
= & -\sum_{p=0}^{m-1}\sum_{l=0}^{p}\binom{n_1}{l}\binom{n_2}{p-l}\phi_{N_1-pm+p,\ n_2+l-p}(x,y)\frac{\partial^{p+1}}{\partial x^{l}\partial y^{p-l+1}}\varphi_m(x,y),
\end{aligned}$$

$$-\sum_{p=0}^{m-1}\sum_{l=0}^{p}\binom{n_1}{l}\binom{n_2}{p-l}\phi_{N_1-pm+p,\ n_2+l-p}(x,y)\frac{\partial^{p+1}}{\partial x^{l+1}\partial y^{p-l}}\varphi_m(x,y)$$

$$= \frac{\partial}{\partial x}\phi_{N_1,\ n_2}(x,y)-\phi_{N_1,\ n_2}(x,y)\frac{\partial}{\partial x}\varphi_m(x,y),$$

$$-\sum_{p=0}^{m-1}\sum_{l=0}^{p}\binom{n_1}{l}\binom{n_2}{p-l}\phi_{N_1-pm+p,\ n_2+l-p}(x,y)\frac{\partial^{p+1}}{\partial x^{l}\partial y^{p-l+1}}\varphi_m(x,y)$$

$$= \frac{\partial}{\partial y}\phi_{N_1,\ n_2}(x,y)-\phi_{N_1,\ n_2}(x,y)\frac{\partial}{\partial y}\varphi_m(x,y)$$

for

$$n_1\geq l,\ n_2\geq p-l,\ m\geq 1.$$

$$\frac{\partial}{\partial x}\phi_{N_1,\ n_2}(x,y)=\phi_{N_1+m-1,\ n_2}(x,y)+\phi_{N_1,\ n_2}(x,y)\frac{\partial}{\partial x}\varphi_m(x,y),$$

$$\frac{\partial}{\partial y}\phi_{N_1,\ n_2}(x,y)=\phi_{N_1+m-1,\ n_2+1}(x,y)+\phi_{N_1,\ n_2}(x,y)\frac{\partial}{\partial y}\varphi_m(x,y)$$

for

$$n_1,\ n_2\geq 0.$$

4. Orthogonality of Special Cases of the Polynomials $\phi_{N,n_2}(x,y)$

In this section, taking some special cases of the polynomials $\phi_{N,\ n_2}(x,y)$, we give orthogonality of the polynomials expressed as product of two classical Hermite polynomials or Laguerre polynomials. Setting $\psi_k(x,y)=xy$ and $\varphi_m(x,y)=x^2+y^2$ in (1.2), we have

$$\begin{aligned}\phi_{n_1+n_2+2,n_2}(x,y) &= e^{x^2+y^2}\frac{\partial^{n_1+n_2}}{\partial x^{n_1}\partial y^{n_2}}\left\{xye^{-x^2-y^2}\right\}\\ &= \left(-\frac{1}{2}e^{x^2}\frac{d^{n_1+1}}{dx^{n_1+1}}e^{-x^2}\right)\left(-\frac{1}{2}e^{y^2}\frac{d^{n_2+1}}{dy^{n_2+1}}e^{-y^2}\right)\\ &= \frac{(-1)^{n_1+n_2}}{4}H_{n_1+1}(x)\,H_{n_2+1}(y)\ ;\ n_1,n_2=0,1,\ldots\end{aligned}$$

where $H_{n_1+1}(x)$ and $H_{n_2+1}(y)$ are Hermite polynomials of degree n_1+1 and n_2+1, respectively.

$$\{F_{n_1,n_2}(x,y)\}=\{\phi_{n_1+n_2+2,n_2}(x,y)\}\ ;\ n_1,n_2=0,1,..$$

polynomials are orthogonal over the domain

$$D=\{(x,y):-\infty<x,y<\infty\ \}$$

with respect to the weight function $\rho(x,y)=e^{-x^2-y^2}$. Infact, from ([6], p.193), we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2} F_{n_1,n_2}(x,y)\, F_{m_1,m_2}(x,y)\, dxdy$$

$$= 2^{n_1+n_2-2}(n_1+1)!\,(n_2+1)!\pi\delta_{n_1,m_1}\delta_{n_2,m_2}$$

where δ_{n_1,m_1} and δ_{n_2,m_2} are Kronecker delta. On the other hand, getting $\varphi_m(x,y) = x^2+y^2+\alpha x$ in (3.5), we have

$$\begin{aligned}
\phi_{n_1+n_2,n_2}(x,y) &= e^{x^2+y^2+\alpha x}\frac{\partial^{n_1+n_2}}{\partial x^{n_1}\partial y^{n_2}}\left\{e^{-x^2-y^2-\alpha x}\right\} \\
&= \left(e^{x^2+\alpha x}\frac{\partial^{n_1}}{\partial x^{n_1}}e^{-x^2-\alpha x}\right)\left(e^{y^2}\frac{\partial^{n_2}}{\partial y^{n_2}}e^{-y^2}\right) \\
&= \left(e^{\left(x+\frac{\alpha}{2}\right)^2}\frac{\partial^{n_1}}{\partial x^{n_1}}e^{-\left(x+\frac{\alpha}{2}\right)^2}\right)\left(e^{y^2}\frac{\partial^{n_2}}{\partial y^{n_2}}e^{-y^2}\right) \\
&= (-1)^{n_1+n_2} H_{n_1}\left(x+\frac{\alpha}{2}\right) H_{n_2}(y);\quad n_1,n_2 = 0,1,\ldots
\end{aligned}$$

where $H_{n_1}\left(x+\frac{\alpha}{2}\right)$ and $H_{n_2}(y)$ are Hermite polynomials of degree n_1 and n_2, respectively. Similarly

$$\{K_{n_1,n_2}(x,y)\} = \{\phi_{n_1+n_2,n_2}(x,y)\}\ ;\ n_1,n_2 = 0,1,..$$

polynomials are orthogonal with respect to the weight function $\rho(x,y) = e^{-\left(x+\frac{\alpha}{2}\right)^2-y^2}$over the domain $D = \{(x,y) : -\infty < x,y < \infty\ \}$. Thus,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left(x+\frac{\alpha}{2}\right)^2-y^2} K_{n_1,n_2}(x,y)\, K_{m_1,m_2}(x,y)\, dxdy$$

$$= 2^{n_1+n_2}(n_1)!\,(n_2)!\pi\delta_{n_1,m_1}\delta_{n_2,m_2}$$

where δ_{n_1,m_1} and δ_{n_2,m_2} are Kronecker delta.

If we take $\psi_k(x,y) = x^{n_1}y^{n_2}$ and $\varphi_m(x,y) = x+y$ in (1.2), we get

$$\begin{aligned}
\phi_{n_1+n_2,n_2}(x,y) &= e^{x+y}\frac{\partial^{n_1+n_2}}{\partial x^{n_1}\partial y^{n_2}}\left\{x^{n_1}y^{n_2}e^{-x-y}\right\} \\
&= \left(e^{x}\frac{\partial^{n_1}}{\partial x^{n_1}}x^{n_1}e^{-x}\right)\left(e^{y}\frac{\partial^{n_2}}{\partial y^{n_2}}y^{n_2}e^{-y}\right) \\
&= n_1!n_2!L_{n_1}(x)\, L_{n_2}(y);\quad n_1,n_2 = 0,1,\ldots
\end{aligned}$$

where $L_{n_1}(x)$ and $L_{n_2}(y)$ are Laguerre polynomials of degree n_1 and n_2, respectively. Therefore,

$$\{K_{n_1,n_2}(x,y)\} = \{\phi_{n_1+n_2,n_2}(x,y)\}\ ;\ n_1,n_2 = 0,1,\ldots$$

polynomials satisfy the following orthogonality relation from ([6] ,p.206):

$$\int_0^{\infty}\int_0^{\infty} e^{-x-y} K_{n_1,n_2}(x,y)\, K_{m_1,m_2}(x,y)\, dxdy = (n_1!)^2(n_2!)^2\delta_{n_1,m_1}\delta_{n_2,m_2}.$$

5. Bilinear and Bilateral Generating Functions

In this section, we derive several families of bilinear and bilateral generating functions for the polynomials $\phi_{N,\ n_2}\ (x,y)$ which are generated by (2.1) and given by (1.2).

We begin by stating the following theorem.

Theorem 5.1. Corresponding to an identically non-vanishing function $\Omega_\mu(y_1,\ldots,y_s\,)$ of s complex variables $y_1,\ldots,y_s$ ($s\in\mathbb{N}$) and of complex order μ, let

$$\Lambda_{\mu,\nu}(y_1,\ldots,y_s;z):=\sum_{l=0}^{\infty}a_l\Omega_{\mu+\nu l}(y_1,\ldots,y_s\,)z^l \tag{5.1}$$

$$(a_l\neq 0\,,\ \ \mu,\nu\in\mathbb{C}).$$

and

$$\begin{aligned}&\Theta_{n_1,n_2,p,\mu,\nu}\,(x_1,x_2;y_1,\ldots,y_s;\varsigma)\\ :&=\sum_{l=0}^{[n_1/p]}\frac{a_l}{(n_1-pl)!}\phi_{N-pl(m-1),\ n_2}\ (x_1,x_2)\,\Omega_{\mu+\nu l}(y_1,\ldots,y_s\,)\varsigma^l\\ &(n_1,p\in\mathbb{N})\,.\end{aligned} \tag{5.2}$$

Then we have

$$\begin{aligned}&\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{1}{n_2!}\Theta_{n_1,n_2,p,\mu,\nu}\left(x_1,x_2;y_1,\ldots,y_s;\frac{\eta}{r_1^p}\right)r_1^{n_1}r_2^{n_2}\\ &=\ \Lambda_{\mu,\nu}(y_1,\ldots,y_s;\eta)\psi_k\,(x_1+r_1,x_2+r_2)\,e^{\varphi_m(x_1,x_2)-\varphi_m(x_1+r_1,x_2+r_2)}\end{aligned} \tag{5.3}$$

provided that each member of (5.3) exists.

Proof. For convenience, let S denote the first member of the assertion (5.3) of Theorem 5.1. Then, upon substituting for the polynomials

$$\Theta_{n_1,n_2,p,\mu,\nu}\left(x_1,x_2;y_1,\ldots,y_s;\frac{\eta}{r_1^p}\right)$$

from the definition (5.2) into the left-hand side of (5.3), we obtain

$$\begin{aligned}S=&\sum_{n_1,n_2=0}^{\infty}\sum_{l=0}^{[n_1/p]}\frac{a_l}{(n_1-pl)!n_2!}\phi_{N-pl(m-1),\ n_2}\,\cdot\\ &\qquad\cdot(x_1,x_2)\,\Omega_{\mu+\nu l}(y_1,\ldots,y_s\,)r_1^{n_1-pl}r_2^{n_2}\eta^l=\\ &=\sum_{l=0}^{\infty}a_l\Omega_{\mu+\nu l}(y_1,\ldots,y_s\,)\eta^l\sum_{n_1,n_2=0}^{\infty}\phi_{N,\ n_2}\ (x_1,x_2)\,\frac{r_1^{n_1}r_2^{n_2}}{n_1!n_2!}=\\ &=\Lambda_{\mu,\nu}(y_1,\ldots,y_s;\eta)\psi_k\,(x_1+r_1,x_2+r_2)\,e^{\varphi_m(x_1,x_2)-\varphi_m(x_1+r_1,x_2+r_2)},\end{aligned}$$

which completes the proof of Theorem 5.1.

In a similar manner, we can prove the following result.

Theorem 5.2. Corresponding to an identically non-vanishing function $\Psi_{M,k_2}(y_1,\ldots,y_s)$ of s complex variables $y_1,\ldots,y_s$ $(s\in\mathbb{N})$ and let

$$\begin{aligned}&\Xi_{\mu,\nu_1,\nu_2}(y_1,\ldots,y_s\,;z,w)\\ :&=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}a_{k_1,k_2}\Psi_{M,k_2}(y_1,\ldots,y_s)z^{k_1}w^{k_2}\end{aligned}\tag{5.4}$$

$$\left(a_{k_1,k_2}\neq 0,\ M=k+(m-1)\left(k_1\nu_1+k_2\nu_2+\mu\right)\ ,\ \mu,\nu_1,\nu_2\in\mathbb{C}\right).$$

and

$$\begin{aligned}&\Phi^{\mu,\nu_1,\nu_2}_{n_1,n_2,p,t}(x_1,x_2;y_1,\ldots,y_s;\zeta,\xi)\\ :&=\sum_{k_1=0}^{[n_1/p]}\sum_{k_2=0}^{[n_2/t]}\frac{a_{k_1,k_2}}{(n_1-pk_1)!\,(n_2-tk_2)!}\phi_{N-(pk_1+tk_2)(m-1),\ n_2-tk_2}\left(x_1,x_2\right)\\ &\times\Psi_{M,k_2}(y_1,\ldots,y_s)\ \zeta^{k_1}\xi^{k_2}\end{aligned}\tag{5.5}$$

where $n_1,n_2,p,t\in\mathbb{N}$. Then

$$\begin{aligned}&\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\Phi^{\mu,\nu_1,\nu_2}_{n_1,n_2,p,t}\left(x_1,x_2;y_1,\ldots,y_s;\frac{\eta}{r_1^p},\frac{\lambda}{r_2^t}\right)r_1^{n_1}r_2^{n_2}\\ =&\Xi_{\mu,\nu_1,\nu_2}(y_1,\ldots,y_s\,;\eta,\lambda)\psi_k\left(x_1+r_1,x_2+r_2\right)e^{\varphi_m(x_1,x_2)-\varphi_m(x_1+r_1,x_2+r_2)}\end{aligned}\tag{5.6}$$

provided that each member of (5.6) exists.

6. Further Consequences and Miscellaneous Properties

By expressing the multivariable function

$$\Omega_{\mu+\nu l}(y_1,\ldots,y_s)\ \left(l\in\mathbb{N}_0\,,\ s\in\mathbb{N}\right)$$

in terms of simpler function of one and more variables, we can give further applications of Theorem 5.1. For example, if we set

$$s=r\ \text{ and }\ \Omega_{\mu+\nu l}(y_1,\ldots,y_r)=h^{(\gamma_1,\ldots,\gamma_r)}_{\mu+\nu l}(y_1,\ldots,y_r)$$

in Theorem 5.1, where the multivariable Lagrange-Hermite polynomials $h^{(\alpha_1,\ldots,\alpha_r)}_n(x_1,\ldots,x_r)$ [2] are generated by

$$\prod_{j=1}^{r}\left\{\left(1-x_jt^j\right)^{-\alpha_j}\right\}=\sum_{n=0}^{\infty}h^{(\alpha_1,\ldots,\alpha_r)}_n\left(x_1,\ldots,x_r\right)t^n\tag{6.1}$$

where $|t|<\min\left\{|x_1|^{-1},|x_2|^{-1/2},\ldots,|x_r|^{-1/r}\right\}$, then we obtain the following result which provides a class of bilateral generating functions for the Lagrange-Hermite multivariable polynomials and the polynomials $\phi_{N,\ n_2}\left(x_1,x_2\right)$ defined by (1.2).

Corollary 6.1. If $\Lambda_{\mu,\nu}(y_1,\ldots,y_r;z):=\sum\limits_{l=0}^{\infty}a_l h_{\mu+\nu l}^{(\gamma_1,\ldots,\gamma_r)}(y_1,\ldots,y_r)z^l$ where $a_l\neq 0\,,\ \ \nu,\mu\in\mathbb{C}$; and

$$\begin{aligned}&\Theta_{n_1,n_2,p,\mu,\nu}\left(x_1,x_2;y_1,\ldots,y_r;\varsigma\right)\\ &:\ =\sum_{l=0}^{[n_1/p]}\frac{a_l}{(n_1-pl)!}\phi_{N-pl(m-1),\ n_2}\left(x_1,x_2\right)h_{\mu+\nu l}^{(\gamma_1,\ldots,\gamma_r)}(y_1,\ldots,y_r)\varsigma^l\end{aligned}$$

where $n_1,p\in\mathbb{N}$. Then we have

$$\begin{aligned}&\sum_{n_1,n_2=0}^{\infty}\frac{1}{n_2!}\Theta_{n_1,n_2,p,\mu,\nu}\left(x_1,x_2;y_1,\ldots,y_r;\frac{\eta}{r_1^p}\right)r_1^{n_1}r_2^{n_2}\\ (6.2)=\ \ &\Lambda_{\mu,\nu}(y_1,\ldots,y_r;\eta)\psi_k\left(x_1+r_1,x_2+r_2\right)e^{\varphi_m(x_1,x_2)-\varphi_m(x_1+r_1,x_2+r_2)}\end{aligned}$$

provided that each member of (6.2) exists.

Remark 6.1. Using the generating function (6.1) and taking $a_l=1$, $\mu=0$, $\nu=1$, we have

$$\begin{aligned}&\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{l=0}^{[n_1/p]}\frac{1}{(n_1-pl)!n_2!}\phi_{N-pl(m-1),\ n_2}\left(x_1,x_2\right)r_2^{n_2}h_l^{(\gamma_1,\ldots,\gamma_r)}(y_1,\ldots,y_r)\\ &\times\eta^l r_1^{n_1-pl}\\ =\ \ &\psi_k\left(x_1+r_1,x_2+r_2\right)e^{\varphi_m(x_1,x_2)-\varphi_m(x_1+r_1,x_2+r_2)}\prod_{j=1}^{r}\left\{\left(1-y_j\eta^j\right)^{-\gamma_j}\right\}\end{aligned}$$

where

$$\left(|\eta|<\min\left\{|y_1|^{-1},|y_2|^{-1/2},\ldots,|y_r|^{-1/r}\right\}\right).$$

Choosing $s=2$ and $\Psi_{M,k_2}(y_1,y_2)=\phi_{M,\ k_2}(y_1,y_2)$ in Theorem 5.2, we obtain the following class of bilinear generating functions for the polynomials $\phi_{N,\ n_2}\left(x_1,x_2\right)$.

Corollary 6.2. If

$$\begin{aligned}&\Xi_{\mu,\nu_1,\nu_2}(y_1,y_2\,;z,w)\\ &:\ =\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}a_{k_1,k_2}\phi_{M,k_2}(y_1,y_2)z^{k_1}w^{k_2}\end{aligned}$$

where $a_{k_1,k_2}\neq 0$, $M=k+(m-1)\left(k_1\nu_1+k_2\nu_2+\mu\right)$, $\mu,\nu_1,\nu_2\in\mathbb{N}_0$ and

$$\begin{aligned}&\Phi_{n_1,n_2,p,t}^{\mu,\nu_1,\nu_2}(x_1,x_2;y_1,y_2;\zeta,\xi)\\ &:\ =\sum_{k_1=0}^{[n_1/p]}\sum_{k_2=0}^{[n_2/t]}\frac{a_{k_1,k_2}}{(n_1-pk_1)!\,(n_2-tk_2)!}\phi_{N-(pk_1+tk_2)(m-1),\ n_2-tk_2}\left(x_1,x_2\right)\\ &\times\phi_{M,\ k_2}(y_1,y_2)\zeta^{k_1}\xi^{k_2}\end{aligned}$$

where $n_1, n_2, p, t \in \mathbb{N}$. Then we have

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\Phi_{n_1,n_2,p,t}^{\mu,\nu_1,\nu_2}\left(x_1,x_2;y_1,y_2;\frac{\eta}{r_1^p},\frac{\lambda}{r_2^t}\right)r_1^{n_1}r_2^{n_2}$$

$$(6.3) = \Xi_{\mu,\nu_1,\nu_2}(y_1,y_2\,;\eta,\lambda)\psi_k\left(x_1+r_1,x_2+r_2\right)e^{\varphi_m(x_1,x_2)-\varphi_m(x_1+r_1,x_2+r_2)}$$

provided that each member of (6.3) exists.

Remark 6.2. Using (2.1) and taking $a_{k_1,k_2} = \dfrac{1}{k_1!k_2!}$, $\mu = 0$, $\nu_1 = \nu_2 = 1$, we have

$$\begin{aligned}&\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{k_1=0}^{[n_1/p]}\sum_{k_2=0}^{[n_2/t]}A^{(p,t)}\left(k_1,k_2,n_1,n_2\right)\phi_{N-(pk_1+tk_2)(m-1),\ n_2-tk_2}\left(x_1,x_2\right)\\&\times\phi_{k+(m-1)(k_1+k_2),\ k_2}(y_1,y_2\,)\eta^{k_1}\lambda^{k_2}r_1^{n_1-pk_1}r_2^{n_2-tk_2}\\=\ &\psi_k\left(x_1+r_1,x_2+r_2\right)\psi_k\left(y_1+\eta,y_2+\lambda\right)e^{\varphi_m(x_1,x_2)-\varphi_m(x_1+r_1,x_2+r_2)}\\&\times e^{\varphi_m(y_1,y_2)-\varphi_m(y_1+\eta,y_2+\lambda)}\end{aligned}$$

where

$$A^{(p,t)}\left(k_1,k_2,n_1,n_2\right) = \frac{1}{k_1!k_2!\left(n_1-pk_1\right)!\left(n_2-tk_2\right)!}.$$

Furthermore, for every suitable choice of the coefficients a_l $(l \in \mathbb{N}_0)$, if the multivariable function $\Omega_{\mu+\nu l}(y_1,\ldots,y_s)$, $(s \in \mathbb{N})$, is expressed as an appropriate product of several simpler functions, the assertions of Theorem 5.1 can be applied in order to derive various families of multilateral generating functions for the polynomials $\phi_{N,\ n_2}$.

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